at the current point. The search direction defined this way is derivative-free. After the calculation of search direction, we modify the line search method and the projection method in [14] to generate a candidate iterative point. Since this point maybe not belong to the constraint set C, the new iterative point is defined as the projection of this candidate point to C, which guarantees that each iterative point belongs to C. The detailed algorithm is described as follows.

Algorithm 2.1 (A Riemannian modified projection method)

Step 0. Give an initial point $p_0 \in \mathcal{H}$, $\sigma > 0$, $\eta > 0$, $\rho \in (0, 1)$, $\nu \in [0, 1)$, $\mu > 0$, and two sequences $\{\nu_k \in [-\nu, \nu]\}$ and $\{\mu_k \geq \mu\}$. Set $k := 0$.
Step 1. If $\|V(p_k)\| = 0$, then stop. Otherwise, go to **Step 2**.
Step 2. Set

$$d_k := \begin{cases} -V(p_0), & k = 0, \\ -V(p_k) + \beta_k \exp_{p_k}^{-1} p_{k-1}, & k > 0, \end{cases} \tag{5}$$

where

$$\beta_k := \frac{\nu_k \|V(p_k)\|^2}{\tau_k + \mu_k \|V(p_k)\|}, \quad \tau_k := \max\left\{|\langle \exp_{p_k}^{-1} p_{k-1}, V(p_k)\rangle|, |\langle \exp_{p_{k-1}}^{-1} p_k, V(p_{k-1})\rangle|\right\}. \tag{6}$$

Step 3. Define the geodesic

$$\gamma_k(t) := \exp_{p_k}(t d_k). \tag{7}$$

Determine $\alpha_k := \eta\rho^{l_k}$ with l_k being the smallest nonnegative integer such that

$$-\langle V(\gamma_k(\alpha_k)), \gamma_k'(\alpha_k)\rangle \geq \sigma\alpha_k \|V(p_k)\|^2. \tag{8}$$

Set

$$q_k := \gamma_k(\alpha_k) = \exp_{p_k}(\alpha_k d_k). \tag{9}$$

Step 4. Define

$$L_k := \{p \in \mathcal{H} \mid \langle \exp_{q_k}^{-1} p, V(q_k)\rangle \leq 0\}. \tag{10}$$

Compute

$$w_k := \Pi_{L_k}(p_k) \tag{11}$$

and

$$p_{k+1} := \Pi_C(w_k). \tag{12}$$

Step 5. Replace k by $k + 1$ and go to **Step 1**.

Remark 2.2 In Algorithm 2.1, $\Pi_{L_k}(p_k)$ denotes the metric projection of p_k onto the set L_k. If the Hadamard manifold $\mathcal{H}$ is of constant curvature, then the halfspace L_k defined by (10) is a closed convex subset of $\mathcal{H}$ [14, Corollary 3.1]. Then the metric projection operator $\Pi_{L_k} : \mathcal{H} \to L_k$ is well-defined. The exact form of $\Pi_{L_k} : \mathcal{H} \to L_k$ depends on the underlying Hadamard manifold $\mathcal{H}$, which is a key step for the implementation of Algorithm 2.1.

Specially, if $\nu = 0$ and $\mathcal{C} = \mathcal{H}$, then $\beta_k = 0$ for $k \geq 0$ and Problem (3) becomes unconstrained. In this case, Algorithm 2.1 reduces to the extragradient method in